Electrohydrodynamic Kelvin-Helmholtz Instability of Two Rotating Dielectric Fluids

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A linear stability analysis of a novel electrohydrodynamic Kelvin-Helmholtz system consisting of the superposition of two uniformly rotating dielectric media is presented. The characteristic equation for such an arrangement is derived, which in turn yields a stability criterion for velocity differences of disturbances at a given rotation frequency. The conditions of stability for long and short wave perturbations are obtained, and their dependence on rotation, surface tension and applied electric field is discussed. Limiting cases for vanishing fluid velocities, rotation frequency, and applied electric field are also discussed. Under suitable limits, results of previous works are recovered. A detailed analysis for tangential and normal applied electric fields, in the presence and absence of surface charges, is carried out.

Key words: Hydrodynamic Stability, Electrohydrodynamics, Interfacial Instability, Rotational Flows, Dielectric Fluids.

1. Introduction

The Kelvin-Helmholtz instability arises when two uniform fluids, separated by a horizontal boundary, are in relative motion. Because of its relevance to astrophysical, geophysical and laboratory situations, this problem has been analysed by several authors [1]. Without surface tension, this streaming is unstable no matter how small the velocity difference between the layers may be. It was shown by Kelvin [2] that surface tension will suppress the instability if the difference in velocity is sufficiently small. The effect of rotation on the instability of two superposed fluids, which has an application to planetary atmospheres, was considered by Hide [3] and Chandrasekhar [4]. It has been shown that the effect of rotation is essentially dependent on the density distribution in the fluid. In two media of uniform density, rotation causes instability, while in a medium of exponentially varying density, rotation has a stabilizing effect.

The effect of uniform rotation about the vertical axis on the development of the Kelvin-Helmholtz instability has been treated by Chandrasekhar [4] and Alterman [5]. Their results show that, in a rotating system, both surface tension and related velocities fail to stabilize the motion for any angular velocity of rotation. For recent works concerning the effect of rotation on surface wave instability problems, see Artale and Salusti [6], Lu and Benney [7], Guillopé et al. [8], Obied Allah [9], Dávalos-Orozco and Aguilar-Rosas [10], Dávalos-Orozco [11], Mehta and Bhatia [12], and Lushchik [13]. For excellent reviews see Drazin and Reid [14], and Saffman [15]. In all the mentioned studies, the effect of electric field, which occurs in astrophysics, geophysics, chemical engineering and industry, see Melcher [16], Haus and Melcher [17] has been neglected.

The mechanical stability of the interfaces in immiscible fluid-fluid systems is a subject of increasing interest. Theoretical models were developed to take into account more accurately the physical and chemical properties of interfaces. Fluid-fluid interfaces can also exhibit electrical charges. External constraints can be imposed, for instance external electric charges, a non-equilibrium density of charges, or an imposed electric current [17]. Scattered literature exists, dating back to the nineteenth century, on a variety of electrofluid phenomena. However, only recently the term "electrohydrodynamics" has come into usage to describe electrofluid interactions between two fluids [16]. Electrohydrodynamics (EHD) studies the interplay of mechanical and electrical forces in fluids. In a first approximation it is assumed that the electri-

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cal currents are very weak and therefore magnetic effects are negligible. Maxwell's equations are then reduced to Gauss's law and the charge conservation law. Within the past few years, a number of papers have appeared on studies of surface stability in the presence of electric fields [18] - [25].

Only few trials have been made to study the effect of electric field on the stability of rotating fluids. For example, El-Dib and Moatimid [18, 19] studied the effect of a periodic rotation and uniform axial and perpendicular electric fields on the stability of two dielectric inviscid fluids separated by a cylindrical interface using a multiple time scales. Mohamed et al. [23] studied the electrohydrodynamic stability of a rotating dielectric jet bounded by an infinite dielectric rotating fluid. Takashima [26] considered the effect of uniform rotation on the onset of convective instability in a dielectric fluid confined between two horizontal planes under the simultaneous action of a vertical ac electric field and a vertical temperature gradient. Oliveri et al. [27] examined the effect of rotation on the EHD stability of a plane layer of fluid subjected to unipolar injection of charge.

In this paper, we consider the electrohydrodynamic stability of two rotating dielectric media which are in relative horizontal motion. The system is influenced by a constant applied electric field tangential, or normal, to the interface between the two media. To the best of my knowledge, the combined effect of applied electric field and uniform rotation on the Kelvin-Helmholtz instability problem (for plane dielectric fluids) has not been investigated yet.

2. The Problem

Let two uniform dielectric fluids of densities $\rho^{(1)}$ and $\rho^{(2)} (< \rho^{(1)})$, dielectric constants $\epsilon^{(1)}$ and $\epsilon^{(2)}$, be separated by a horizontal boundary at $z = 0$. Suppose also that the fluids are streaming with constant velocities $U^{(1)}$ and $U^{(2)}$, where the superscripts (1) and (2) refer to the lower and the upper fluid, respectively. We shall further suppose that the fluids are in uniform rotation about the z-axis with an angular velocity Ω, and the system be subjected to a constant tangential electric field E_0 in the x-direction.

The equations governing the motion are [5]

$$\rho\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}\right] = -\boldsymbol{\nabla} p - 2\rho(\boldsymbol{\Omega}\times\boldsymbol{v}) - \rho\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}) - \rho g\boldsymbol{k} + \boldsymbol{F},$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0,$$

where $\boldsymbol{F}$ is the external applied field, and since

$$\boldsymbol{\Omega}\times\boldsymbol{v} = -\Omega v\boldsymbol{i} + \Omega u\boldsymbol{j},$$

$$\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}) = -\Omega^2 x\boldsymbol{i} - \Omega^2 y\boldsymbol{j},$$

the former equations reduce to

$$\rho\left[\frac{\partial u}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})u - 2\Omega v\right] = -\frac{\partial \Pi}{\partial x}, \quad (1)$$

$$\rho\left[\frac{\partial v}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})v + 2\Omega u\right] = -\frac{\partial \Pi}{\partial y}, \quad (2)$$

$$\rho\left[\frac{\partial w}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})w\right] = -\frac{\partial \Pi}{\partial z} - \rho g, \quad (3)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (4)$$

where the total pressure is defined as

$$\Pi = p - \frac{1}{2}\rho\Omega^2 r^2 - \frac{1}{2}\epsilon E_0^2, \quad (5)$$

where p is the pressure, ϵ the permittivity, $\boldsymbol{E}$ the electric field, $\boldsymbol{v}$ the fluid velocity, and $r^2 = x^2 + y^2$.

We also assume that the quasi-static approximation is valid for the problem, and hence the electric field can be derived from a scalar potential ψ. Accordingly, Maxwell's equations reduce to [16]

$$\boldsymbol{\nabla}\cdot(\epsilon\boldsymbol{E}) = 0 \quad \text{and} \quad \boldsymbol{\nabla}\times\boldsymbol{E} = 0, \quad (6)$$

where

$$\boldsymbol{E} = -\boldsymbol{\nabla}\psi \quad \text{and} \quad \boldsymbol{\nabla}^2\psi = 0. \quad (7)$$

3. Perturbation Equations and Solutions

As a result of a disturbance, let the components of the velocity in the perturbed state be $(U + u_1)$, v_1, w_1, and the corresponding increments in $\boldsymbol{E}$ and Π be $\boldsymbol{E}_1$ and Π_1. The linearization of (1) - (4), (6) and (7) leads to the perturbation equations

$$\rho\left[\frac{\partial u_1}{\partial t}+U\frac{\partial u_1}{\partial x}-2\Omega v_1\right] = -\frac{\partial \Pi_1}{\partial x},$$
$$\rho\left[\frac{\partial v_1}{\partial t}+U\frac{\partial v_1}{\partial x}+2\Omega u_1\right] = -\frac{\partial \Pi_1}{\partial y},$$
$$\rho\left[\frac{\partial w_1}{\partial t}+U\frac{\partial w_1}{\partial x}\right] = -\frac{\partial \Pi_1}{\partial z},$$
$$\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}+\frac{\partial w_1}{\partial z} = 0,$$

and

$$\boldsymbol{E}_1 = -\boldsymbol{\nabla}\psi_1 \quad \text{and} \quad \boldsymbol{\nabla}^2\psi_1 = 0.$$

Analysing the disturbance in terms of normal modes, we seek solutions whose dependence on the horizontal x and y axes and on time are given by

$$e^{i(k_x x+k_y y+nt)}. \tag{8}$$

Under these assumptions the above perturbation equations can be written in the form

$$i\rho(n+k_xU)u_1-2\rho\Omega v_1 = -ik_x\Pi_1, \tag{9}$$
$$i\rho(n+k_xU)v_1+2\rho\Omega u_1 = -ik_y\Pi_1, \tag{10}$$
$$i\rho(n+k_xU)w_1 = -D\Pi_1, \tag{11}$$
$$\text{where } D = \mathrm{d}/\mathrm{d}z,$$
$$k_xu_1+k_yv_1 = iDw_1, \tag{12}$$
$$(D^2-k^2)\psi_1 = 0. \tag{13}$$

Multiplying (9) and (10) by $-ik_x$ and $-ik_y$, respectively, adding and making use of (12), we obtain

$$i\rho(n+k_xU)Dw_1+2\rho\Omega\xi = -k^2\Pi_1$$

where $\xi = ik_xv_1-ik_yu_1$ is the z-component of the vorticity $\boldsymbol{\nabla}\times\boldsymbol{v}$. Also, multiplying (9) and (10) by $-ik_y$ and ik_x, respectively, adding and making use (12), we get

$$\xi = -\frac{2i\Omega}{(n+k_xU)}Dw_1.$$

Substitution of ξ into the preceding equation yields

$$\Pi_1 = -\frac{i\rho(n+k_xU)}{k^2}\left[1-\frac{4\Omega^2}{(n+k_xU)^2}\right]Dw_1, \tag{14}$$

also, substituting (14) into (11), we obtain

$$(D^2-K^2)w_1 = 0, \tag{15}$$

where

$$K_j = k\left[1-\frac{4\Omega^2}{(n+k_xU^{(j)})^2}\right]^{-1/2}, \; j=1,2. \tag{16}$$

The solutions of (13) and (15) are

$$w_1^{(j)} = A_je^{i(k_xx+k_yy+nt)\pm K_jz}, \tag{17}$$
$$\psi_1^{(j)} = B_je^{i(k_xx+k_yy+nt)\pm kz}, \; j=1,2. \tag{18}$$

Now, we define the elevation of the interface from the equilibrium state as

$$\zeta = \delta e^{i(k_xx+k_yy+nt)}, \tag{19}$$

where δ is a small parameter. The unit normal $\boldsymbol{N}$ to the interface $F = z-\zeta$, to the first order terms, is

$$\boldsymbol{N} = \boldsymbol{\nabla}F/|\boldsymbol{\nabla}F| = -ik_x\zeta\boldsymbol{i}-ik_y\zeta\boldsymbol{j}+\boldsymbol{k},$$

and the total electric field in the two regions is given by

$$\boldsymbol{E}^{(j)} = E_0\boldsymbol{i}-\boldsymbol{\nabla}\psi_1^{(j)}, \; j=1,2$$

for the lower and upper fluids, respectively.

4. Stability of Disturbances

To determine the constants A_j, B_j, $j=1,2$ in (17) and (18), we use the following boundary conditions which should be satisfied at the interface between the two media:

(1) The kinematic boundary condition is satisfied at $z=0$, i. e.

$$w_1^{(j)} = \frac{\partial\zeta}{\partial t}+U^{(j)}\frac{\partial\zeta}{\partial x}, \; j=1,2. \tag{20}$$

Hence, we obtain

$$A_j = i\delta(n+k_xU^{(j)}), \; j=1,2.$$

Therefore, (17) can be written as

$$w_1^{(j)} = i\zeta(n+k_xU^{(j)})e^{\pm K_jz}. \tag{21}$$

Substituting (21) into (14), we obtain

$$\Pi_1^{(j)} = \pm \frac{\zeta \rho^{(j)} (n + k_x U^{(j)})^2}{k} \cdot \left[1 - \frac{4\Omega^2}{(n + k_x U^{(j)})^2} \right] e^{\pm K_j z}. \tag{22}$$

(2) The tangential component of the electric field should be continuous at the interface, i. e.

$$\frac{\partial \psi_1^{(1)}}{\partial x} = \frac{\partial \psi_1^{(2)}}{\partial x} \text{ at } z = 0. \tag{23}$$

(3) The normal component of the electric displacement is continuous at the interface, i. e.

$$\boldsymbol{N} \cdot \epsilon^{(1)} \boldsymbol{E}^{(1)} = \boldsymbol{N} \cdot \epsilon^{(2)} \boldsymbol{E}^{(2)} \text{ at } z = 0. \tag{24}$$

Using the expressions for $\boldsymbol{N}$ and $\boldsymbol{E}^{(2)}$, $j = 1, 2$, therefore, (18), (23) and (24) lead to

$$\psi_1^{(j)} = \frac{i k_x E_0 (\epsilon^{(2)} - \epsilon^{(1)}) \zeta}{k (\epsilon^{(2)} + \epsilon^{(1)})} e^{\pm kz}. \tag{25}$$

(4) The normal component of the stress tensor is discontinuous at the interface by the surface tension, i. e.

$$\Pi_{ij}^{(2)} - \Pi_{ij}^{(1)} = -T \boldsymbol{\nabla}^2 \zeta \text{ at } z = \zeta, \tag{26}$$

where the stress tensor is given by [16, 28]

$$\Pi_{ij} = -\Pi \delta_{ij} + \epsilon E_i E_j - \frac{1}{2} \epsilon E^2 \delta_{ij}.$$

Equation (26), to the first order terms, can be written in the form

$$\zeta \frac{\partial \Pi_0^{(1)}}{\partial z} - \zeta \frac{\partial \Pi_0^{(2)}}{\partial z} + \Pi_1^{(1)} - \Pi_1^{(2)} - \epsilon^{(1)} E_0 \frac{\partial \psi_1^{(1)}}{\partial x} + \epsilon^{(2)} E_0 \frac{\partial \psi_1^{(2)}}{\partial x} = -T \boldsymbol{\nabla}^2 \zeta \text{ at } z = 0, \tag{27}$$

where the equilibrium state $\Pi_0^{(j)}$ can be expressed from the basic equation of motion (3) in the form

$$\frac{\partial \Pi_0^{(j)}}{\partial z} = -\rho^{(j)} g, \; j = 1, 2. \tag{28}$$

Substituting (19), (22), (25) and (28) into (27), and letting

$$\alpha^{(j)} = \frac{\rho^{(j)}}{(\rho^{(2)} + \rho^{(1)})}, \; j = 1, 2, \tag{29}$$

the characteristic equation for this arrangement will be given by

$$\alpha^{(1)} (n + k_x U^{(1)})^2 \left[1 - \frac{4\Omega^2}{(n + k_x U^{(1)})^2} \right]^{1/2} + \alpha^{(2)} (n + k_x U^{(2)})^2 \left[1 - \frac{4\Omega^2}{(n + k_x U^{(2)})^2} \right]^{1/2} - \frac{k_x^2 E_0^2 (\epsilon^{(2)} - \epsilon^{(1)})^2}{(\rho^{(2)} + \rho^{(1)})(\epsilon^{(2)} + \epsilon^{(1)})} = gk \left[(\alpha^{(1)} - \alpha^{(2)}) + \frac{T k^2}{g (\rho^{(2)} + \rho^{(1)})} \right]. \tag{30}$$

For perturbations in the direction perpendicular to the direction of the velocity and electric field vectors, the frequency $k_x = 0$ and $k_y = k$. In this case, (30) gives

$$n^2 \left[1 - \frac{4\Omega^2}{n^2} \right]^{1/2} = gk \left[(\alpha^{(1)} - \alpha^{(2)}) + \frac{T k^2}{g (\rho^{(2)} + \rho^{(1)})} \right],$$

which is the equation for Rayleigh-Taylor instability in a rotating fluid obtained by Chandrasekhar [4]. In this equation, neither the horizontal velocity of the fluid in each region nor the applied electric field has an effect on the perturbation.

The electrohydrodynamic Kelvin-Helmholtz instability is least uninhibited for perturbations in the direction of streaming. Let us further consider the case $k_x = k$. Then (30) becomes

$$\alpha^{(1)} (n + kU^{(1)})^2 \left[1 - \frac{4\Omega^2}{(n + kU^{(1)})^2} \right]^{1/2} + \alpha^{(2)} (n + kU^{(2)})^2 \left[1 - \frac{4\Omega^{(2)}}{(n + kU^{(2)})^2} \right]^{1/2} - \frac{k^2 E_0^2 (\epsilon^{(2)} - \epsilon^{(1)})^2}{(\rho^{(2)} + \rho^{(1)})(\epsilon^{(2)} + \epsilon^{(1)})} = gk \left[(\alpha^{(1)} - \alpha^{(2)}) + \frac{T k^2}{g (\rho^{(2)} + \rho^{(1)})} \right], \tag{31}$$

which reduces to the equation obtained by Alterman [5] if the applied electric field is absent. With the notations

$$c = -\frac{n}{k}, \quad (U^{(1)} - c) = \xi\sqrt{\frac{g}{k}}, \quad (U^{(2)} - c) = \eta\sqrt{\frac{g}{k}}, \quad \omega^2 = \frac{4\Omega^2}{gk} \quad \text{and} \quad \Theta = \frac{k^2 T}{g(\rho^{(1)} - \rho^{(2)})}$$

(31) reduces to

$$\frac{\gamma_1}{(1+\gamma)}\xi^2\sqrt{1 - \frac{\omega^2}{\xi^2}} + \frac{\gamma_2}{(1+\gamma)}\eta^2\sqrt{1 - \frac{\omega^2}{\eta^2}} = 1 \tag{32}$$

with

$$\gamma_j = \frac{\alpha^{(j)}}{(\alpha^{(1)} - \alpha^{(2)})(1+\Theta)} \quad \text{and} \quad \gamma = \frac{E_0^2(\epsilon^{(2)} - \epsilon^{(1)})^2}{(\epsilon^{(2)} + \epsilon^{(1)})}\,\frac{\Theta}{kT(1+\Theta)}\,, \; j = 1, 2.$$

Equation (32) defines a real locus in the (ξ, η) plane. The part of the locus in the first quadrant connects the points (ω, η_0) and (ξ_0, ω), where

$$(\xi_0, \eta_0) = \left[\frac{\omega^2}{2} + \frac{1}{2}\sqrt{\omega^4 + \frac{4(1+\gamma)^2}{\gamma_{1,2}^2}}\right]^{1/2}.$$

The analysis given by Chandrasekhar [4] applies to (32), and as a result there will be stability for

$$|U^{(1)} - U^{(2)}|\sqrt{\frac{k}{g}} \le \eta_0 - \omega,$$

i. e.

$$\frac{1}{2\Omega}|U^{(1)} - U^{(2)}| \le \frac{1}{k}\left[\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + (Ak + Ck^2 + Bk^3)^2}\right)^{1/2} - 1\right] = L(k), \tag{33}$$

where

$$A = \frac{g(\alpha^{(1)} - \alpha^{(2)})}{2\alpha^{(2)}\Omega^2}\,, \quad B = \frac{T}{2\rho^{(2)}\Omega^2} \quad \text{and} \quad C = \frac{E_0^2(\epsilon^{(2)} - \epsilon^{(1)})^2}{2\rho^{(2)}\Omega^2(\epsilon^{(2)} + \epsilon^{(1)})}. \tag{34}$$

Equation (33) gives the velocity differences for stability of disturbances for any given fixed frequency k. It shows the stabilizing effect of the tangential electric field.

5. Long and Short Wave Perturbations

For long wave disturbances, $k \to 0$, then (33) and (34) lead to

$$L(k) \to \frac{1}{k}\left[\sqrt{\frac{1}{2} + \frac{1}{2}\left(1 + \frac{1}{2}(Ak + Ck^2 + Bk^3)^2\right)} - 1\right]$$

$$\to \frac{A^2}{8}k$$

or $\quad |U^{(1)} - U^{(2)}| \le A^2 k\Omega/4.$

This relation can be simplified to the form

$$|U^{(1)} - U^{(2)}| \le \frac{g^2 k(\alpha^{(1)} - \alpha^{(2)})^2}{16\,\alpha^{(2)^2}\,\Omega^3}. \tag{35}$$

This result is independent of surface tension and electric field effects and can be stated as follows:

- For any given difference $(U^{(1)} - U^{(2)})$ and angular frequency Ω, there are perturbations of such low frequency as to cause instability.

Therefore, surface tension and tangential electric field have no effect on the stability criterion for such long waves and the predominant effect in (35) is due to rotation.

For short wave perturbations, $k \to \infty$ and $\Omega \neq 0$, then (33) and (34) lead to

$$L(k) \to \sqrt{\frac{1}{2k^2} + \frac{1}{2}\sqrt{\frac{1}{k^4} + \left(\frac{A}{k} + C + Bk\right)^2} - \frac{1}{k}} \to \frac{1}{2}\sqrt{2(C + Bk)}$$

or

$$|U^{(1)} - U^{(2)}| \leq \Omega\sqrt{2(C + Bk)},$$

which can be written in the form

$$|U^{(1)} - U^{(2)}|^2 \leq \frac{Tk}{\rho^{(2)}} + \frac{E_0^2(\epsilon^{(2)} - \epsilon^{(1)})^2}{\rho^{(2)}(\epsilon^{(2)} + \epsilon^{(1)})}. \quad (36)$$

Equation (36) indicates that the rotation has no effect on the stability criterion for such short waves. In the presence of both surface tension and electric field, the coefficients B and C are different from zero, and from (36), the arrangement will be stable for short wavelengths. It is clear that both the surface tension and the tangential electric field has a stabilizing effect in this case. The effect of the electric field is to increase the surface tension by an amount $E_0^2(\epsilon^{(2)} - \epsilon^{(1)})^2/\rho^{(2)}(\epsilon^{(2)} + \epsilon^{(1)})$. Clearly, the surface tension T has the predominant stabilizing effect for short wave perturbations, and the tangential electric field adds to it.

Without surface tension and electric field effects, $B = C = 0$. In this case, for any $(U^{(1)} - U^{(2)})$ and Ω there will be short waves which cause instability. In other words, in the absence of both surface tension and electric field, both very long and very short wave perturbations cause instability. Both surface tension and tangential electric field stabilize the arrangement only for short wave perturbations, as to be expected.

6. Some Limiting Cases

Now, we shall discuss some limiting cases of interest for the characteristic equation (31) as follows:

(i) In the absence of streaming, i. e. when $U^{(1)} = U^{(2)} = 0$, equation (31) leads to

$$n^4 - 4\Omega^2 n^2 - n_0^4 = 0, \quad (37)$$

where

$$n_0^2 = \frac{k}{(\rho^{(1)} + \rho^{(2)})} \cdot \left[Tk^2 + \frac{kE_0^2(\epsilon^{(2)} - \epsilon^{(1)})^2}{(\epsilon^{(2)} + \epsilon^{(1)})} + g(\rho^{(1)} - \rho^{(2)})\right]. \quad (38)$$

The solution of (37) is

$$n^2 = 2\Omega^2 \pm \sqrt{4\Omega^4 + n_0^4} \text{ if } n_0^2 > 0 \text{ and } n_0^2 < 0. \quad (39)$$

From (39) it follows that, in the absence of streaming, the presence of rotation does not affect the instability or stability, as such, of a stratification: $n^2 > 0$ if $n_0^2 > 0$ and $n^2 < 0$ as $n_0^2 < 0$. Therefore stability is possible if $n_0^2 > 0$, which is the same condition for stability as given by Melcher [16], and Chandrasekhar [4] if the electric field is absent.

On the other hand, according to (39)

$$n^2 \longrightarrow 4\Omega^2 \text{ if } n_0^2 \longrightarrow 0 \quad \text{for positive values.} \quad (40)$$

The frequency of the stable oscillations can not be greater than the natural frequency of wave propagation in a rotating fluid.

(ii) In the absence of rotation, i. e. when $\Omega = 0$, (31) leads to

$$n^2 + 2nk(\alpha^{(1)}U^{(1)} + \alpha^{(2)}U^{(2)}) + \left[k^2(\alpha^{(1)}U^{(1)^2} + \alpha^{(2)}U^{(2)^2}) - n_0^2\right] = 0,$$

whose solution is given by

$$n = -k(\alpha^{(1)}U^{(1)} + \alpha^{(2)}U^{(2)}) \pm \sqrt{n_0^2 - \alpha^{(1)}\alpha^{(2)}k^2(U^{(1)} - U^{(2)})^2}, \quad (41)$$

where the + and $-$ signs in equation (41) correspond to the cases when $n_0^2 > \alpha^{(1)}\alpha^{(2)}k^2(U^{(1)} - U^{(2)})^2$ and $n_0^2 < \alpha^{(1)}\alpha^{(2)}k^2(U^{(1)} - U^{(2)})^2$, respectively. From (41) it follows that $n > 0$ if $n_0^2 > \alpha^{(1)}\alpha^{(2)}k^2(U^{(1)} - U^{(2)})^2$ and $n < 0$ if $n_0^2 < \alpha^{(1)}\alpha^{(2)}k^2(U^{(1)} - U^{(2)})^2$.

Therefore, stability is possible if

$$(U^{(1)} - U^{(2)})^2 < \frac{(\rho^{(1)} + \rho^{(2)})}{\rho^{(1)}\rho^{(2)}}\left[Tk + \frac{g}{k}(\rho^{(1)} - \rho^{(2)}) + \frac{E_0^2(\epsilon^{(2)} - \epsilon^{(1)})^2}{(\epsilon^{(2)} + \epsilon^{(1)})}\right]; \quad (42)$$

otherwise, the system is unstable.

Now, if $\rho^{(1)} > \rho^{(2)}$ the right hand side of (42) is positive. The configuration shall be unstable if the square of the relative velocity exceeds a certain critical value defined by (42) for the equality sign. If, however, $\rho^{(1)} < \rho^{(2)}$ and also

$$Tk + \frac{E_0^2(\epsilon^{(2)} - \epsilon^{(1)})^2}{(\epsilon^{(2)} + \epsilon^{(1)})} < \frac{g}{k}(\rho^{(2)} - \rho^{(1)}),$$

there shall be instability for any finite value of relative tangential velocity. But if

$$Tk + \frac{E_0^2(\epsilon^{(2)} - \epsilon^{(1)})^2}{(\epsilon^{(2)} + \epsilon^{(1)})} > \frac{g}{k}(\rho^{(2)} - \rho^{(1)})$$

for the case $\rho^{(1)} < \rho^{(2)}$, there can be stability provided that

$$(U^{(1)} - U^{(2)})^2 < \frac{(\rho^{(1)} + \rho^{(2)})}{\rho^{(1)}\rho^{(2)}} \left[Tk + \frac{E_0^2(\epsilon^{(2)} - \epsilon^{(1)})^2}{(\epsilon^{(2)} + \epsilon^{(1)})} - \frac{g}{k}(\rho^{(2)} - \rho^{(1)})\right]$$

for any particular value of the wavenumber k.

If $\rho^{(1)} = \rho^{(2)} = \rho$ the condition for stability or instability is written as

$$(U^{(1)} - U^{(2)})^2 < \frac{2E_0^2(\epsilon^{(2)} - \epsilon^{(1)})^2}{\rho(\epsilon^{(2)} + \epsilon^{(1)})}$$

or

$$(U^{(1)} - U^{(2)})^2 > \frac{2E_0^2(\epsilon^{(2)} - \epsilon^{(1)})^2}{\rho(\epsilon^{(2)} + \epsilon^{(1)})},$$

which shows that a vortex sheet alone (i. e. $E_0 = 0$) is unstable and the presence of a horizontal electric field improves the stability of the vortex sheet [29].

(iii) In the absence of the electric field, i. e. when $E_0 = 0$ we obtain the equation obtained earlier by Chandrasekhar [4] for Kelvin-Helmholtz instability in the presence of rotation. Also, as explained before, the stability condition for the velocity differences of the fluids will be the same as given by (35) (which corresponds to gravity waves) in the case of long wave perturbations, which indicates the destabilizing effect of the rotation, and for short wave disturbances, the stability condition for the velocity differences (36) will take the form

$$|U^{(1)} - U^{(2)}| \le \sqrt{\frac{Tk}{\rho^{(2)}}},$$

which corresponds to capillary waves and indicates that the surface tension has a stabilizing influence, but no effect of the rotation on the stability criterion.

(iv) In the absence of both fluid velocities and rotation, i. e. when $U^{(1)} = U^{(2)} = \Omega = 0$, (31) reduces to the form

$$n^2 = n_0^2 = \frac{k}{(\rho^{(1)} + \rho^{(2)})} \cdot \left[Tk^2 + \frac{kE_0^2(\epsilon^{(2)} - \epsilon^{(1)})^2}{(\epsilon^{(2)} + \epsilon^{(1)})} + g(\rho^{(1)} - \rho^{(2)})\right].$$

The case $n^2 = 0$ corresponds to the neutral or marginal state separating stable and unstable regions. The linear electrohydrodynamic Rayleigh-Taylor instability for a tangential electric field is governed by the above dispersion relation [16]. For stability, the condition $n^2 > 0$ is satisfied, implying that the electric field is strictly stabilizing in the linear sense regardless of which of the two fluids has a larger dielectric constant. Therefore the system is stable if the wavenumber k exceeds a critical value k_c given by

$$k = \sqrt{\frac{g}{T}(\rho^{(2)} - \rho^{(1)})}(\cosh\theta - \sinh\theta),$$

where

$$\sinh\theta = \frac{E_0^2(\epsilon^{(2)} - \epsilon^{(1)})^2}{2(\epsilon^{(2)} + \epsilon^{(1)})\sqrt{gT(\rho^{(2)} - \rho^{(1)})}}$$

which is the result obtained earlier by Mohamed and Elshehawey [24].

7. Effect of Normal Electric Field

Here, we consider the electrohydrodynamic Kelvin-Helmholtz instability of two superposed semi-infinite dielectric media, under the actions of uniform rotation with an angular velocity Ω, about the vertical z-axis, and constant applied electric fields $E_0^{(1)}$ and $E_0^{(2)}$ in the negative z-direction. The total electric field in this case will be given by

$$\boldsymbol{E}^{(j)} = -\boldsymbol{E}_0^{(j)}\boldsymbol{k} - \boldsymbol{\nabla}\psi_1^{(j)}, \quad j = 1, 2.$$

We shall distinguish between the following two cases.

(1) When there are no surface charges at the interface

In this case, the condition $\epsilon^{(1)}E_0^{(1)} = \epsilon^{(2)}E_0^{(2)}$ will be satisfied [16], and the solutions of the problem will be given by (21) and (22) together with

$$\psi_1^{(j)} = \mp \frac{\zeta E_0^{(j)}(\epsilon^{(2)} - \epsilon^{(1)})}{(\epsilon^{(2)} + \epsilon^{(1)})} e^{\pm kz}, \; j = 1, 2. \tag{43}$$

Following the preceding procedure the characteristic equation will be given by

$$\alpha^{(1)}(n + k_x U^{(1)})^2 \sqrt{1 - \frac{4\Omega^2}{(n + k_x U^{(1)})^2}} + \alpha^{(2)}(n + k_x U^{(2)})^2 \sqrt{1 - \frac{4\Omega^2}{(n + k_x U^{(2)})^2}} + \frac{k^2 E_0^{(1)} E_0^{(2)} (\epsilon^{(2)} - \epsilon^{(1)})^2}{(\rho^{(2)} + \rho^{(1)})(\epsilon^{(2)} + \epsilon^{(1)})}$$
$$= gk \left[(\alpha^{(1)} - \alpha^{(2)}) + \frac{Tk^2}{g(\rho^{(2)} + \rho^{(1)})} \right]. \tag{44}$$

If $k_x = 0$, $k_y = k$, (44) reduces to

$$n^2 \sqrt{1 - \frac{4\Omega^2}{n^2}} + \frac{k}{(\rho^{(1)} + \rho^{(2)})} \left[\frac{k E_0^{(1)} E_0^{(2)} (\epsilon^{(2)} - \epsilon^{(1)})^2}{(\epsilon^{(2)} + \epsilon^{(1)})} - \left\{ g(\rho^{(1)} - \rho^{(2)}) + Tk^2 \right\} \right] = 0,$$

which represents the dispersion equation for the electrohydrodynamic Rayleigh-Taylor instability in a rotating fluid influenced by normal electric fields. It can be written in the form

$$n^4 - 4\Omega^2 n^2 - N_0^4 = 0, \tag{45}$$

where

$$N_0^2 = \frac{k}{(\rho^{(2)} + \rho^{(1)})} \left[\frac{k E_0^{(1)} E_0^{(2)} (\epsilon^{(2)} - \epsilon^{(1)})^2}{(\epsilon^{(2)} + \epsilon^{(1)})} - \left\{ g(\rho^{(1)} - \rho^{(2)}) + Tk^2 \right\} \right]. \tag{46}$$

The solution of (45) is given by

$$n^2 = 2\Omega^2 \pm \sqrt{4\Omega^4 - N_0^4} \text{ if } N_0^2 > 0 \text{ and } N_0^2 < 0,$$

as mentioned before, in the limiting case (i) of Section 6. Therefore, stability is possible if $N_0^2 > 0$, i. e. if the condition

$$Tk^2 - \frac{k E_0^{(1)} E_0^{(2)} (\epsilon^{(2)} - \epsilon^{(1)})^2}{(\epsilon^{(2)} + \epsilon^{(1)})} + g(\rho^{(1)} - \rho^{(2)}) < 0$$

is satisfied, which indicates that the rotation has no effect on the stability criterion.

If we consider only perturbations in the direction of streaming, i. e. $k_x = k$, then (44) reduces to

$$\alpha^{(1)}(n + kU^{(1)})^2 \sqrt{1 - \frac{4\Omega^2}{(n + kU^{(1)})^2}} + \alpha^{(2)}(n + kU^{(2)})^2 \sqrt{1 - \frac{4\Omega^2}{(n + kU^{(2)})^2}} + N_0^2 = 0.$$

This equation, as in the case of a tangential electric field, can be written as

$$\frac{\gamma_1}{(1 - \gamma^*)} \xi^2 \sqrt{1 - \frac{\omega^2}{\xi^2}} + \frac{\gamma_2}{(1 - \gamma^*)} \eta^2 \sqrt{1 - \frac{\omega^2}{\eta^2}} = 1, \tag{47}$$

where $\gamma_{1,2}$ and Θ are defined in Sect. 4, and

$$\gamma^* = \frac{E_0^{(1)} E_0^{(2)} (\epsilon^{(2)} - \epsilon^{(1)})^2}{(\epsilon^{(2)} + \epsilon^{(1)})} \frac{\Theta}{kT(1 + \Theta)}.$$

Thus, there will be stability for

$$\frac{1}{2\Omega} |U^{(1)} - U^{(2)}| \le \frac{1}{k} \left[\sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{1 + (Ak - C_1 k^2 + Bk^3)^2}} - 1 \right], \tag{48}$$

which gives the velocity differences for stability of disturbances for any given fixed frequency k, where A and B are defined in Sect. 4, and

$$C_1 = \frac{E_0^{(1)} E_0^{(2)} (\epsilon^{(2)} - \epsilon^{(1)})^2}{2\rho^{(2)} \Omega^2 (\epsilon^{(2)} + \epsilon^{(1)})}.$$

Relation (48) shows the destabilizing influence of the normal applied electric field.

For long wave disturbances we obtain the result given by (35), which indicates that the normal electric field also has no effect on the stability criterion for such long waves. For short wave perturbations, we have

$$|U^{(1)} - U^{(2)}|^2 \leq \frac{Tk}{\rho^{(2)}} - \frac{E_0^{(1)} E_0^{(2)} (\epsilon^{(2)} - \epsilon^{(1)})^2}{\rho^{(2)} (\epsilon^{(2)} + \epsilon^{(1)})}, \quad (49)$$

which shows that the electric field has a destabilizing influence, where the electric field is to decrease the surface tension by an amount $E_0^{(1)} E_0^{(2)} (\epsilon^{(2)} - \epsilon^{(1)})^2 / \rho^{(2)} (\epsilon^{(2)} + \epsilon^{(1)})$. In this case the rotation has no effect on the stability criterion for such short waves.

(2) When there are surface charges at the interface

The condition given in the previous case can not be satisfied here, i. e. we have the condition $\epsilon^{(1)} E_0^{(1)} \neq \epsilon^{(2)} E_0^{(2)}$ [16]. In this case, the boundary condition that the normal component of the electric displacement is continuous at the interface should be replaced by the condition that the tangential component of the stress tensor is continuous at the interface. The corresponding solutions of the problem will be given by (21) and (22) together with

$$\psi_1^{(1)} = -\zeta E_0^{(j)} e^{\pm kz}, \quad j = 1, 2, \quad (50)$$

and the characteristic equation will be given by

$$\begin{aligned} &\alpha^{(1)} (n + k_x U^{(1)})^2 \sqrt{1 - \frac{4\Omega^2}{(n + k_x U^{(1)})^2}} \\ &+ \alpha^{(2)} (n + k_x U^{(2)})^2 \sqrt{1 - \frac{4\Omega^2}{(n + k_x U^{(2)})^2}} \\ &+ \frac{k^2}{(\rho^{(2)} + \rho^{(1)})} \left\{ \epsilon^{(1)} E_0^{(1)^2} + \epsilon^{(2)} E_0^{(2)^2} \right\} \\ &\qquad = gk \left[(\alpha^{(1)} - \alpha^{(2)}) + \frac{Tk^2}{g(\rho^{(2)} + \rho^{(1)})} \right]. \end{aligned} \quad (51)$$

As in the previous case, putting $k_x = k$ and comparing (51) and (44), we can write the velocity differences for stability of disturbances for any fixed frequency k in the form

$$\frac{1}{2\Omega} |U^{(1)} - U^{(2)}| \leq \left[\sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{1 + (Ak - C_2 k^2 + Bk^3)^2}} - 1 \right], \quad (52)$$

where

$$C_2 = \frac{1}{2\rho^{(2)} \Omega^2} \left\{ \epsilon^{(1)} E_0^{(1)^2} + \epsilon^{(2)} E_0^{(2)^2} \right\}$$

Equation (52) shows also the strong destabilizing influence of the normal applied electric field in this case.

Thus, replacing $E_0^{(1)} E_0^{(2)} (\epsilon^{(2)} - \epsilon^{(1)})^2 / (\epsilon^{(2)} + \epsilon^{(1)})$ by $(\epsilon^{(1)} E_0^{(1)^2} + \epsilon^{(2)} E_0^{(2)^2})$ of case (**1**) where there are no surface charges at the interface, one gets quite similar results. The effect of the normal electric field if there are surface charges at the interface between the two media, has more destabilizing effect than in case (**1**), i. e. it has a strong destabilizing influence in the short wave limit owing to the form of the electric field term.

8. Conclusions

We have investigated the linear electrohydrodynamic Kelvin-Helmholtz instability of two superposed dielectric fluids in case of uniform rotation and the presence of an electric field. The results of this study can be summerized as follows:

(**1**) For the horizontal electric fields and horizontal perturbations we obtained the stability condition for the velocity difference of the fluids. This stability condition for long wave perturbations is inversely proportional to the cube of the angular frequency, and both the surface tension and the applied electric field have no effect on the stability. For short wave disturbances, the stability condition yields that both the surface tension and the applied electric field have a stabilizing effect, and that the rotation has no effect on the stability.

(**2**) For vertical electric fields and in the absence of surface charges at the interface between the two fluids, the stability conditions are obtained and discussed for perturbations in the vertical direction. For horizontal perturbations we obtained the velocity differences for the stability at any frequency. It is found that for long wave perturbations the results are similar to those for horizontal electric fields.

For short wave disturbances we found that the electric field is destabilizing and the surface tension is stabilizing, while the rotation has no effect on the stability.

In the presence of charges at the interface between the two fluids, it is found that similar results as in the case of absence of surface charges are obtained, when replacing the term $E_0^2(\epsilon^{(2)} - \epsilon^{(1)})^2/(\epsilon^{(2)} + \epsilon^{(1)})$ by the term $(\epsilon^{(1)} E_0^{(1)^2} + \epsilon^{(2)} E_0^{(2)^2})$. It is found also that the vertical electric field has more destabilizing effect than the horizontal one.

Finally, the special cases for the absence of streaming in the fluids, rotation, and applied electric fields are studied. They are found to be in exact agreement with the corresponding previous works in fluid mechanics and electrohydrodynamics.

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